

# From circular paths to elliptic orbits: A geometric approach to Kepler's motion

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## Abstract

The hodograph, i.e. the path traced by a body in velocity space, was introduced by Hamilton in 1846 as an alternative method for studying certain dynamical problems. The hodograph of the Kepler problem was then investigated and shown to be a circle, it was next used to investigate some other properties of the motion. We here propose a new method for tracing the hodograph and the corresponding configuration space orbit in Kepler's problem starting from the initial conditions given and trying to use no more than the methods of synthetic geometry in a sort of Newtonian approach. All of our geometric constructions require straight edge and compass only.

## Resumen

La hodógrafa, i.e. la curva recorrida por un cuerpo en el espacio de las velocidades, fué propuesta por Hamilton en 1846 como una alternativa para investigar algunos problemas dinámicos. Se demostró entonces que la hodógrafa del problema de Kepler es una circunferencia y posteriormente se la usó para establecer algunas otras propiedades del movimiento. En este trabajo proponemos un método geométrico semi newtoniano para construir una órbita elíptica partiendo de sus condiciones iniciales y de la correspondiente hodógrafa, empleando para ello métodos de la geometría sintética que requieren de la regla y del compás únicamente.

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## 1. Introduction

The Kepler problem has a great deal to offer whenever different approaches to its solution are sought (Rosales et al 1991, Sivardi re 1992, Mart nez-y-Romero et al 1993); this follows since it is a member of the very limited class of superintegrable system (Evans 1990, Mart nez-y-Romero et al 1992, Salas-Brito et al 1997). One of the simpler approaches to solve the problem starts by constructing its hodograph (Gonz lez-Villanueva et al 1996, 1998a,b); let us recall that the hodograph is just the path traced by the velocity of a body as function of time. In this work we aim to go from Kepler’s hodograph to the corresponding configuration space orbit in a geometric quasi-Newtonian fashion. Besides the intrinsic beauty of geometrical arguments, we think our approach can contribute to a better understanding of the interrelations between the initial conditions and the properties of the Kepler problem solutions.

The hodograph was introduced by Hamilton in the XIX century as an alternative for solving dynamical problems; perhaps the greatest triumph of the method was in the analysis of Kepler motions (Hamilton 1846, Maxwell 1877, Thomson and Tait 1879). However, even if one discovers, as Hamilton did, that the hodograph is circular in such a case, it is natural to wonder how can it be related to the conic section orbits. The problem is easily solved in the analytical treatment (Gonz lez-Villanueva et al 1996, 1998a) but there also exist beautiful geometric approaches for finding such relation developed by Maxwell (1877) and by Feynman (Goodstein and Goodstein 1996). In this work we want to explain our own version of the geometric relationship between the hodograph and the orbit, but we understand that this approach ‘cannot substantially differ from what any other [...] student can construct’ (Chandrasekhar 1995, p xxiii).

## 2. Kepler’s hodograph is circular

To establish the circularity of the Kepler hodograph, let us start from the equation of motion of a particle in a fixed Newtonian field at the origin

$$m \frac{d\mathbf{v}}{dt} = -\frac{GMm}{r^2} \hat{\mathbf{e}}_r, \quad (1)$$

where  $m$ ,  $M$ ,  $G$ ,  $\mathbf{v}$  and  $\hat{\mathbf{e}}_r$  are, respectively, the mass of the particle, the mass of the attracting body, the Newtonian gravitational constant, the velocity and the unit radial vector. As in any central problem, the energy  $E$  and the angular momentum  $\mathbf{L} = m\mathbf{r} \times \mathbf{v} = mr^2\dot{\theta}\hat{\mathbf{e}}_z = L\hat{\mathbf{e}}_z$  are conserved —the  $\mathbf{L}$ -direction defines our  $z$ -axis. Please note our convention that the trajectory in configuration space is the *orbit* whereas the trajectory in velocity space is the *hodograph*. The Kepler orbits are thus confined to a plane orthogonal to  $\mathbf{L}$  which includes the origin; a polar coordinate system with unit vectors  $\hat{\mathbf{e}}_r$  and  $\hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_r$  is chosen in this plane. Moreover, just from the polar identities  $\dot{\hat{\mathbf{e}}}_r = \dot{\theta}\hat{\mathbf{e}}_\theta$ , and  $\dot{\hat{\mathbf{e}}}_\theta = -\dot{\theta}\hat{\mathbf{e}}_r$ , it is easy to see that equation (1) can be rewritten as (Moreno 1990)

$$\dot{\mathbf{v}} = \frac{\alpha/m}{r^2} \hat{\mathbf{e}}_r = \frac{\alpha}{L} \dot{\hat{\mathbf{e}}}_\theta, \quad (2)$$

where we introduced the positive constant  $\alpha \equiv GMm$ . From (2), it should be clear that the velocity can be written as (González-Villanueva et al 1996)

$$\mathbf{v} = \mathbf{h} + \frac{\alpha}{L} \hat{\mathbf{e}}_\theta \quad (3)$$

where the Hamilton vector  $\mathbf{h}$  is a *constant of motion*. As (3) makes obvious, the velocity traces a circle with radius  $R_h = \alpha/L$ ; the *hodograph* is thus a circle centered at  $\mathbf{h}$ . Then  $\mathbf{h}$  points along the dynamical symmetry axis of the hodograph—dynamical since it is not only a geometrical property, the interaction intervenes directly; we have found that  $\mathbf{h}$  defines what we call the hodograph’s dynamical diameter, the line  $\overline{XX_s}$  shown in figure 1. By extension, equation (3) also shows that every orbit has a dynamical symmetry axis, which is found geometrically in section 3 below. Notice that  $\overline{XX_s}$  can also serve to construct the notorious Laplace or Runge-Lenz vector  $\mathbf{A} \equiv \mathbf{h} \times \mathbf{L}$  (Landau and Lifshitz 1976), see section 3 and figure 2, which points toward the pericentre of the orbit. In what follows, we measure angles in  $r$ -space counterclockwise from  $\mathbf{A}$ , and in  $v$ -space, from  $\mathbf{h}$ . The circularity of the hodograph implies that the orbit is both periodic and symmetric and that those points on the hodograph where the velocities are antiparallel must be symmetric on the orbit. Notice also that the energy in any Kepler motion can be related to the magnitude of  $\mathbf{h}$  as follows

$$E = \frac{m}{2} (h^2 - \alpha^2/L^2); \quad (4)$$

as can be seen in (4), the bounded or unbounded nature of the motions change according to where the  $v$ -space origin is positioned in relation to the hodograph. It suffices that the  $v$ -origin be within the hodograph (i.e.  $R_h > h$ ) to assure that the hodograph is the whole circle and that the orbit is elliptic (González-Villanueva et al 1996, 1998a). How to draw the orbit in this case once the hodograph is known is discussed below in section 4.

### 3. Drawing the hodograph

To draw the hodograph given the initial position  $\mathbf{r}_0$  and velocity  $\mathbf{v}_0$ , we need to fix the magnitude of the angular momentum  $L = mr_0 v_0 \sin \delta$ , where  $0 \leq \delta \leq \pi$  is the angle between the initial position and velocity (see  $\delta$  in figure 4); we also need to fix the quantity  $R_h = \alpha/L$ . Please keep an eye on figure 1 while reading the following.

Let the point  $F$  be the position of the centre of force (hence, the  $r$ -origin). Draw the line segment  $\overline{FR}$  as the initial position  $\mathbf{r}_0$ ; extend it up to an arbitrary point  $O$ —we are here just choosing the origin in velocity space. From the  $v$ -origin  $O$ , draw the line segment  $\overline{OV}$  corresponding to  $\mathbf{v}_0$  and erect, perpendicular to  $\overline{FR}$ , a line segment  $\overline{OO'}$  of length  $R_h$ —that is, we are drawing  $-\alpha/L \hat{\mathbf{e}}_\theta$  (recall that we defined  $\hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_r$ , where  $\hat{\mathbf{e}}_z \equiv \mathbf{L}/L$ ). Notice that the previous construction assumes an attractive interaction. Now sum  $\overline{OV}$  to  $\overline{OO'}$  to get the point  $C$ . The line segment  $\overline{OC}$ , as follows from equation (3), represents the Hamilton vector. Having obtained  $\mathbf{h}$ , just draw with centre at  $C$  a circle of radius  $R_h$  to get the hodograph. This geometrical procedure, besides giving  $\mathbf{h}$  and the hodograph, informs us about

the energy of the motion. It is only a matter of noticing whether  $O$  is within the hodograph or not; if it is within, the energy is negative, if not, the energy is positive. Figure 1 illustrates a case in which  $O$  is within, that is, a motion with  $E < 0$ ; as follows from equation (4),  $E = 0$  occurs when  $O$  is located precisely on the circle, i.e. when  $h = R_h = \alpha/L$ .

To draw the dynamical symmetry axis of the orbit (i.e. the line on which the Laplace vector  $\mathbf{A}$  lays) from the given initial conditions, just draw the line segment  $\overline{FS}$ , which is perpendicular to  $\overline{OC}$  going through the centre of force  $F$ ; as the segment labeled  $\mathcal{A}$  in figure 2 illustrates. This follows from the parallellism of  $\mathbf{h}$  and the velocity at pericentre  $\mathbf{v}_p$ . The line  $\overline{FS}$  so drawn, is *the dynamical symmetry axis of the orbit*. Notice also that  $\mathbf{v}_p$  can be drawn by prolonguing the segment  $\overline{OC}$  until it intersects the hodograph. This intercept is marked  $X$  in figure 1. If, as happens in figures 1, 2, and 3, there are two intersections with the hodograph and not just one, the velocity space origin  $O$  is necessarily inside the hodograph, that is, it always corresponds to the case  $E < 0$ . The second intercept, labeled  $X_s$  in figure 1, defines the segment  $\overline{OX_s}$  corresponding to the velocity at the *apocentre* of the orbit, i.e., at the point on the orbit farthest from the centre of force where the speed is the lowest possible.

#### 4. How to trace the elliptic orbit

Let us assume that the origin of coordinates in velocity space happens to be inside the circle of the hodograph; this is the case whose realization from initial conditions was discussed in section 2 and illustrated in figure 1. Please refer to figure 2 for the schematic representation of the geometric steps that follow. The points  $F$ ,  $R$ ,  $O$ ,  $O'$ ,  $V$  and  $C$  in figure 2 have exactly the same meaning as in figure 1, that is, they serve to construct the Hamilton vector  $\overline{OC}$  and the hodograph centered at  $C$  given the initial conditions  $\mathbf{r}_0$  (the straight line  $\overline{FR}$ , which makes an angle  $\theta$  with  $\mathcal{A}$ ), and  $\mathbf{v}_0$  (the straight line  $\overline{OV}$ ) and the vector  $-\hat{\mathbf{e}}_\theta R_h$  (the straight line  $\overline{OO'}$ ). The initial velocity also helps to define the segment  $\overline{CV}$  making the same angle  $\theta$  with  $\mathbf{h}$ . In fact, we will always assume this meaning for the labeling of points in figures 2–4.

To locate any point on the orbit, first extend the straight line  $\overline{OV}$  until it again intercepts the hodograph at  $V_s$  (see figure 2). Then trace a perpendicular to  $\overline{CV_s}$  passing through  $R$ , this line intercepts the symmetry axis (drawn as in section 3) at the auxiliar point  $F'$ . To locate the point on the orbit corresponding to any given point on the hodograph, let us first notice that we have already one such pair of points, the initial conditions: point  $R$  and point  $V$ . Let us choose another point  $V'$  on the hodograph, then draw the straight line  $\overline{OV'}$  and extend it until it intersects the hodograph at point  $V'_s$ . Draw two straight lines perpendicular to  $\overline{CV'}$  and to  $\overline{CV'_s}$  passing through  $F$  and  $F'$ , respectively; we assert that these two perpendiculars meet at the required point  $R'$  on the orbit, as was the case with the perpendiculars to the segments  $\overline{CV}$  and  $\overline{CV_s}$ , related to the initial conditions and meeting at  $R$ . To draw the complete orbit, i.e. the gray curve in figure 3, we have to repeat the procedure starting from each point on the hodograph.

To decide the shape of the constructed orbit, draw the circular arc  $F'W$  centered at  $R$  with a radius equal to the length of  $\overline{F'R}$ ; this arc intercepts the straight line  $\overline{FO}$  at the point  $W$  (see figure 3). Next, trace the circular arc  $WW'$  centered at  $F$  with radius  $\overline{FW}$ . It is now easy to see, just by noticing that the shaded triangles  $\triangle V'V'_sC$  and  $\triangle W'F'R'$  are both isosceles and similar to each other, that the point  $R'$  on the orbit is at the same distance from the point  $W'$  than from the point  $F'$ . We can see thus that the radius of the circular arc  $WW'$  is the sum of the lengths of  $\overline{FR'}$  and  $\overline{F'R'}$  and, therefore, that *in the case  $E < 0$  the orbit is necessarily an ellipse* whose major axis  $2a$  equals the length of the line  $\overline{FW}$ . The auxiliary point  $F'$  is thus seen to be the second focus of the ellipse, the first one coinciding with the centre of force  $F$ . The line  $\overline{FS}$  is parallel to the symmetry axis of the ellipse as we had anticipated. In fact, the eccentricity of the ellipse is easily calculated as  $\epsilon = h/R_h = OC/CV$  (González-Villanueva et al 1996). Thus,  $\mathbf{A}$ , is the line segment parallel to  $\overline{FS}$  of length  $\alpha\epsilon$  pointing towards the pericentre. The construction performed here also shows that  $\overline{FR'}$  always makes the same angle with  $\overline{AB}$  than  $\overline{CV'}$  with  $\overline{OC}$  (figures 2 and 3). Geometric methods for reconstructing the orbit in the cases  $E \geq 0$  are described in detail in (González-Villanueva et al 1998c).

## 5. Why the method works

Let us first pinpoint the uniqueness of the elliptic orbit drawn in figure 3; this follows since the initial conditions,  $\mathbf{r}_0$ ,  $\mathbf{v}_0$ , uniquely specify both  $\mathbf{L}$  and  $\mathbf{h}$ . These, in turn, are the necessary and sufficient conditions to obtain  $\mathbf{A}$  and, hence, both the dynamical symmetry axis and the orbit (González-Villanueva et al 1996, 1998b). But, even with the uniqueness established, the relationship of the hodograph with the orbit, and the lines used in section 3, can still remain obscure. How can we ascertain that the velocity at any point on the hodograph is parallel to the tangent at the corresponding point on the ellipse? An attempt to explain the situation is in figure 4, which re-elaborates figure 3 removing certain unnecessary features, and in the explanation that follows:

Let us assume that the construction of section 3 has been carried out. To start the explanation, draw a circle (see figures 3 and 4) with radius  $\overline{FW}$  (i.e. radius  $2a$ ) centered at  $F$  (this corresponds to the arc  $WW'W''$  in figure 3); trace the straight line  $\overline{AB}$ , corresponding to the ellipse's dynamical symmetry axis, on this line mark the second focus  $F'$ . Next, pick an arbitrary point  $R$  on the orbit and trace the segment  $\overline{FR}$  making an angle  $\theta$  with  $\overline{AB}$  and extend it until the intercept  $W$  on the circle. This defines the segments  $\overline{F'W}$  and its continuation  $\overline{F'W_s}$ . Now trace their perpendicular bisectors  $\overline{MR}$  and  $\overline{M_sR_s}$ , these lines intercept  $\overline{FW}$  and  $\overline{FW_s}$  at  $R$  and  $R_s$ , respectively. Extend  $\overline{FW_s}$  until it intercepts the circle at  $W''$  and trace the lines  $\overline{FW''}$  and  $\overline{WW''}$ . Thence the triangles  $\triangle FWW''$ ,  $\triangle RWF'$  and  $\triangle R_sF'W_s$ , are isosceles, similar to each other and with common angles  $\delta$  and  $\pi - 2\delta$ . From this, it is easy to see that  $\overline{W_sR_s} = \overline{R_sF'}$  and  $\overline{F'R} = \overline{RW}$ , i.e. they belong to the same ellipse. Besides, the lines  $\overline{F'R}$  and  $\overline{FR_s}$  are parallel by construction, the same is true of lines  $\overline{R_sF'}$  and  $\overline{FR}$ , therefore  $\overline{FR_s}$  and  $\overline{F'R}$  make the same angle  $\xi$  with  $\overline{AB}$ .

Notice that  $\triangle WW_s W''$  is a right triangle by construction thence,  $\overline{WW''}$  and the perpendicular bisectors  $\overline{MR}$  and  $\overline{M_s R_s}$  are parallel to each other. Notice that the perpendicular bisector to segment  $\overline{WW_s}$  also bisects the angle  $\angle WFW_s$  and therefore that the lines  $\overline{MR}$  and  $\overline{M_s R_s}$  are tangent to the ellipse at  $R$  and at  $R_s$ . This establishes that, in effect, the tangent at every point on the orbit is parallel to the *corresponding* velocity on the hodograph and, at the same time, that every pair of symmetric points on the hodograph, *where the velocities are antiparallel*, corresponds to a pair of symmetric points on the orbit.

It is now easy to see that if we rescale the circle  $WW'W''$  in figure 3, or  $AWW''$  in figure 4, by the factor  $\alpha/2aL$  we get essentially the hodograph but rotated  $\pi/2$  respect that in figure 1; furthermore, under these same rescaling and rotation, the lines  $\overline{F'F}$ ,  $\overline{F'W}$  and  $\overline{F'W_s}$  in figure 4, become, respectively, the Hamilton vector and the velocities at  $V$  and  $V_s$ , all shown in figure 3.

## 6. Concluding remarks

We have shown how the bounded orbits of the Kepler problem can be drawn starting from the initial conditions—and the hodograph—using no more than straight edge, compass and a few lines in a piece of paper. We have also exhibited that the hodograph and  $\mathbf{h}$  are crucial for deciding *geometrically* if the orbits are bounded or not and, furthermore, that with their help, we can draw any orbit starting from arbitrary initial conditions. Although we have not shown it here, these elementary geometrical techniques can be quite useful for discussing orbital manoeuvres and other features of the motion in a Newtonian field. This means that our approach can provide a very convenient method for addressing the interplay between the physics and the geometry of Kepler's problem in a kind of Newtonian fashion.

The need to make the construction presented more accesible has prompted us to program our construction of the orbits using The Geometer's Sketchpad3<sup>©</sup> a very nice piece of software for doing geometrical constructions which can be obtained at [http://www.keypress.com/product\\_info/sketch-demo.html](http://www.keypress.com/product_info/sketch-demo.html) in a demo version. The demonstration of our constructions has been succesful with the students. Any interested reader may try to reproduce the method using the simple language associated with the Sketchpad.

As a final remark, we must say that our main motivation for this work must be found on the amusement side. We have had a lot of fun in trying to do mechanics without using most of the usual analytic techniques of contemporary physics. We hope this article may convey to the readers the sense of enjoyment we discovered in the geometric beauty of dynamics. In our eyes at least—though beauty is in the eyes of the beholder!— these considerations are enough to justify the quasi-Newtonian approach presented in this article. To finalise, we found convenient to paraphrase Chandrasekhar's paraphrasing of Ben Johnson (Chandrasekhar 1995, Epilogue) since it clearly summarizes our viewpoint: *Newton's methods were not of an age, but for all time!*

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## Figure Captions

### Figure 1

The geometrical procedure for obtaining both the Hamilton vector and the hodograph from given initial conditions  $\mathbf{r}_0$  and  $\mathbf{v}_0$  is illustrated.  $O$  labels the origin of coordinates in velocity space or  $v$ -origin and  $F$  labels the location of the centre of force or  $r$ -origin. To draw the segment  $OO'$ , corresponding to  $-\hat{\mathbf{e}}_\theta \alpha/L$ , we assumed that  $\mathbf{L}$  points outside the plane of the paper. The Hamilton vector is the line segment  $\overline{OC}$ , the circle  $X_s V X$  centered at  $C$  is the hodograph. The straight segments  $\overline{SF}$  and  $\overline{SX}$  correspond, respectively, to the dynamical symmetry axes of the orbit and of the hodograph. The discussion related to this figure can be found in section 3.

### Figure 2

The procedure for reconstructing the orbit when the hodograph encompass the  $v$ -origin is illustrated.  $F$ , corresponds to the position of the centre of force,  $O$  is the  $v$ -origin,  $V$  and  $R$  are an arbitrary velocity on the hodograph and its corresponding position on the elliptic orbit, and  $C$  is the geometric centre of the hodograph.  $\mathcal{A}$  represents the Laplace (or Runge-Lenz) vector. For a detailed discussion of the method for reconstructing the orbit see section 4.

### Figure 3

To prove that the orbit is indeed an ellipse (the only case considered in this article), we need the help of an auxiliary circle with radius  $2a$ , equal to the length of  $\overline{W_s W''}$ , and to recognize that the two shaded isosceles triangles  $\triangle R' W' F'$  and  $\triangle C V' V'_s$  are similar to each other.

### Figure 4

This is essentially figure 3 excepting for some details unnecessary for the explanation in section 5. The purpose of this figure is to explain the reasoning behind the method used to construct the orbit starting from the hodograph. The circle  $AWW''$  (with radius  $2a$ ) corresponds to the circle  $WW'W''$  in figure 3 and, after a  $\pi/2$ -rotation and a rescaling by  $\alpha/2aL$ , it also corresponds to the hodograph in figures 1, 2 and 3. It is convenient to remember that a conic can be defined as the locus of points being at the same distance from both a fixed point ( $F'$ ) and a fixed circle (the arc  $WW'W''$ ). The angle between the initial position and the initial velocity is  $\delta$ .